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J. Knott, "Robert Recorde, a pioneer mathematician, astronomer, and physician," *Indian Medical Record*, Calcutta, 1904, vol. 25, pp. 1–3.

D. E. Smith, *Rara Arithmetica*, Boston, 1908, pp. 213–221, 253, 286–288. [Title pages of *The Ground of Artes*, 1558, and *The Whetstone of Witte*, 1557, are here reproduced; also pages from these works showing counter reckoning and the explanation of the use of the sign for equality].

H. Zeitlinger and H. C. S., *Bibliotheca Chemico-Mathematica*, Catalogue 682 of Henry Sotheran & Co., London, 1908, p. 197; (also in *Bibliotheca Chemico-Matematica*, vol. 1, London, 1921, p. 197. There is here, opposite page 200, a facsimile page from *The Castle of Knowledge*, the first English work recognizing the Copernican system.)

L. C. Karpinski, "The Whetstone of Witte (1557)," *Bibliotheca Mathematica*, 1913, series 3, vol. 13, pp. 223–228.

J. Knott, "Robert Recorde," *Nature*, Dec. 7, 1916, vol. 98, p. 268, see also p. 172.

F. P. Barnard, *The Casting-Counter and the Counting-Board. A Chapter in the History of Numismatics and Early Arithmetic.* Oxford, 1916, pp. 254–266.

F. Cajori, *History of Elementary Mathematics with Hints on Methods of Teaching.* Revised and enlarged edition. New York, 1917, pp. 183–188; numerous other page references are given in the index.

F. V. Morley, "Finis coronat opus," *Scientific Monthly*, 1920, vol. 10, 306–308.

ON A DIOPHANTINE PROBLEM.

BY O. D. KELLOGG, Harvard University.

Professor Carmichael has been kind enough to take an interest in, and give some currency to, a problem in Diophantine analysis which I communicated to him some years ago.[1] I stated to him at that time that the maximum value of any of the unknowns that can occur in a solution in positive integers of the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1 \tag{1}$$

was u_n, where $u_1 = 1$, and

$$u_{k+1} = u_k(u_k + 1). \tag{2}$$

His remark, in the review cited, that a complete theory of this equation seemed desirable stimulated me to attempt to reconstruct the proof of my statement, which proof I do not seem to have preserved. The attempt, however, has given rise to a doubt in my mind as to my ever having had a really valid proof, but at the same time to confirm my belief in the accuracy of the statement.

It therefore seems to me proper to make the above confession to such of the readers of the MONTHLY as may have taken an interest in the problem, and to tell what I do know about it, for it has some rather nice aspects. Diophantine problems have at least the conspicuous merit that many phases of them are quite intelligible without profound knowledge of analysis, and for that reason are frequently a stimulus to mathematical interest.

The problem arose in connection with the familiar mapping of the surface of a triangle in the z-plane upon the upper half of the w-plane by means of an analytic function of a complex variable, $w = f(z)$. If the mapping is extended by reflections of the triangle in its sides, and thus gives rise to a single valued function, it must be possible to cover the plane with non-overlapping triangles congruent with the given one. That is, if the angles are $a\pi$, $b\pi$, $c\pi$, then a, b, c must be aliquot parts of unity with sum equal to 1. It was an extension of the question of determining what those angles could be which was formulated in the above problem.

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[1] See his *Diophantine Analysis*, New York, 1915, p. 115; also his review of Dickson's *History of the Theory of Numbers*, vol. 2, in this MONTHLY, *1921*, 77.

In a certain sense a problem may be considered as solved when it is shown how its solutions may be determined by a finite number of trials. For instance, the problem of finding the rational roots of a polynomial equation with integral coefficients may be regarded as settled by the theorem that if m and n are relatively prime, m/n cannot be a root of the equation unless n is a factor of the coefficient of the highest power of the variable present in the equation, and m is a factor of the coefficient of the lowest power. In the case of the problem of finding the positive integral solutions of the equation (1), it is clear that for $n > 1$, 2 is a minimum value for any unknown. Hence all that is needed in order to show that the solutions can be found by a finite number of trials is the determination of a finite upper limit for the numbers which may occur in a solution.

I therefore offer the following "proof" of my statement about the maximum of an x_i. Regarding as identical solutions obtained from each other by a permutation of the values of the x_i, we may fix our attention on those in which the x_i form a monotone increasing sequence. Then x_n will be a maximum when $(1/x_1) + (1/x_2) + \cdots + (1/x_{n-1})$ has "exhausted" 1 to as great a degree as possible, *i.e.*, when $1 - [(1/x_1) + (1/x_2) + \cdots + (1/x_{n-1})]$ has the least possible positive value. The greatest exhaustion of 1 which is possible by one term is obtained by taking $x_1 = 2$. This leaves 1/2. The greatest exhaustion of this 1/2 which is possible by the next term is obtained by taking $x_2 = 3$. The greatest exhaustion of the remaining 1/6 which is possible by the next term is obtained by taking $x_3 = 7$, and so on. In general, if u_k is the maximum value of x_k in the equation (1) with $n = k$, we exhaust 1 best with k terms by leaving $x_1, x_2, x_3, \cdots x_{k-1}$ unchanged, giving to x_k the value $u_k + 1$. This leaves

$$\frac{1}{u_k} - \frac{1}{u_k + 1} = \frac{1}{u_k(u_k + 1)}$$

for the $(k + 1)$th term, *so that* $u_{k+1} = u_k(u_k + 1)$ *is the maximum value of an* x_i *in the equation* (1) *with* $k + 1$ *terms.*

The "proof" is phrased as if entirely valid in order to allow the reader, if so disposed, to criticize it before reading the exposê given now. The logical lapse, of course, consists in the implied assumption that the minimum of a function (here $1 - (1/x_n)$) may be obtained by minimizing it *successively* with respect to each independent variable rather than *simultaneously* with respect to them all.

A few remarks seem worth while. First, it should be observed that the method sketched always gives *a* solution, namely $x_1 = u_1 + 1$, $x_2 = u_2 + 1$, $\cdots$, $x_{n-1} = u_{n-1} + 1$, $x_n = u_n$. Further, while space is lacking to give all the considerations which tend to substantiate the correctness of the assertion about the maximum x_n, one or two are worth mentioning. It will be observed that x_n is not greater than the least common multiple of all the other x_i, as may be seen by subtracting the sum of their reciprocals from 1, and that it is equal to this least common multiple if they are all relatively prime. It therefore appears plausible that the maximum x_n will occur when the other x_i are all relatively prime. But this is a property of the number sequence $u_k + 1$. To show this, I form the difference equation for $v_k = u_k + 1$:

$$v_{k+1} = v_k^2 - v_k + 1. \tag{3}$$

Iteration shows that v_{k+m} is a polynomial in v_k with constant term 1, so that no divisor of v_k can divide v_{k+m}. The first four values of v_k are 2, 3, 7, 43, so that one might be tempted to guess that he had here a sequence of primes. He would be promptly undeceived, however, on finding that the next number, 1807 has the factor 13. Indeed it appears difficult even to decide whether or not the sequence contains an infinite number of primes. It is, however, a simple matter to show that certain primes never occur as factor in any of the v_k, for instance 5 and 11.

That the solution containing the maximum x_n is far from the symmetric one in which all the x_i are equal ($x_i = n$), is evidenced by the following theorem. *In the solution containing the maximum x_n, no two of the x_i can be equal, except possibly the last two.* For, if $x_i = x_{i+1}$, we have the equation in $n - 1$ unknowns

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{2}{x_i} + \frac{1}{x_{i+2}} + \cdots + \frac{1}{x_n} = 1,$$

which, on dividing by 2 and adding 1/2 to both sides, becomes the equation in n unknowns

$$\frac{1}{2} + \frac{1}{2x_1} + \frac{1}{2x_2} + \cdots + \frac{1}{x_i} + \frac{1}{2x_{i+2}} + \cdots + \frac{1}{2x_n} = 1,$$

where x_n has been replaced by a larger integer. Furthermore, in the solution containing the biggest x_n, not even the last two denominators can be equal *and even,* if $n > 2$. For $2/x_n$ can be replaced by the sum with a larger denominator

$$\frac{1}{\frac{x_n}{2} + 1} + \frac{1}{\frac{x_n}{2}\left(\frac{x_n}{2} + 1\right)}.$$

It seems as if the difference equations (2) and (3) might merit study, not only because they are probably the simplest type of non-linear difference equations, but because of some rather interesting arithmetical properties of their solutions. I will only remark that the second gives a rapidly convergent series for unity. It is

$$1 = \frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \cdots, \qquad \text{or} \qquad 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \cdots.$$

Not that it is recommended to use the series for computing the value of its sum approximately, but that Liouville[1] has proved the existence of infinitely many transcendental numbers by setting up rapidly convergent numerical series whose sums are transcendental. The present series does not converge as rapidly as those given by him, but it is more rapidly convergent than the exponential series for $x = 1$, which defines the transcendental number e. In fact v_n lies between $2^{(2^n-2)}$ and $2^{(2^n)}$.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

10. Delambre Aids in Freeing Spencer Stanhope.

Not only was Delambre influential in securing the release of the founder of the Smithsonian Institution from prison in Hamburg, as stated in a preceding article,[2] and thus in an indirect way in the establishing of the Institution itself, but he was of service to British science and art in performing a similar service for Spencer Stanhope, a young member of the famous Stanhope family and one who at that time gave much promise in antiquarian research.

One of Delambre's letters in my collection reads as follows:

> The Perpetual Secretary for mathematical sciences, to His Excellency the Minister of War.
> *Monsieur Le Duc,*
>
> A young Englishman, M. Spencer Stanhope, made a prisoner of war through a series of unhappy circumstances surrounding his peaceful voyage to Greece, has begged the Imperial Institute to interest itself in his favor with respect to Your Excellency.
>
> The research which M. Stanhope is undertaking is concerned particularly with antiquities, and this is the reason which has determined the Class of Ancient Languages to recommend to Your Excellency the [favorable consideration of the] petition of this young traveler. He also includes in his plan the rendering of aid to geography, to astronomy, and to all the sciences; he is supplied with telescopic octants and he proposes in his travels to join certain geographers with whom he expects to work in Greece and the adjoining countries. His undertaking will be of service to practically all Classes and sections of the Imperial Institute. He goes out from England in search of new information which he proposes to take back for the common advantage of all civilized nations. The Imperial Institute, in applauding his designs, believes that it participates in the intentions of a generous government in recommending to Your Excellency the [granting of the] request of M. Stanhope, who earnestly desires that he may be permitted to continue the voyage so unhappily interrupted.
>
> I am, with respect, Monsieur Le Duc,
> Your Excellency's very humble and
> very obedient servant,
> Delambre.

The letter bears an official memorandum of reference to the same official as the one relating to James Smithson, and doubtless was equally successful in accomplishing its purpose.

11. Voltaire and Mathematics.

Of those who admire that greatest champion of popular liberty in the eighteenth century, it is not probable that one in a thousand connects Voltaire's

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[1] *Journal de mathématiques pures et appliquées*, series 1, vol. 16, 1851, pp. 137ff.

[2] No. 1 of this series, *1921*, 64-65.